STUDENT'S NAME: ____________________________________

TEACHER'S NAME: ____________________________________

2024

HURLSTONE AGRICULTURAL HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Task 4 of 4

Mathematics Extension 1

Senior Examiner:

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black pen
- NESA approved calculators may be used
- A reference sheet is provided in the Section I booklet
- For questions in Section II, use the relevant booklet for writing your solutions.
- This examination paper is not to be removed from the examination centre

Total marks: 70

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10. The multiple-choice answer sheet has been provided
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6 – 13)

- Attempt Questions 11 – 15, write your solutions in the spaces provided.
- You have been provided with 5 separate answer booklets, one for each question.
- Extra working pages are available if required.
- Allow about 1 hours and 45 minutes for this section.

SECTION I (10 marks)

Attempt Questions 1 - 10

Allow about 15 minutes on this section.

Use the multiple-choice answer sheet provided for Questions 1 – 10.

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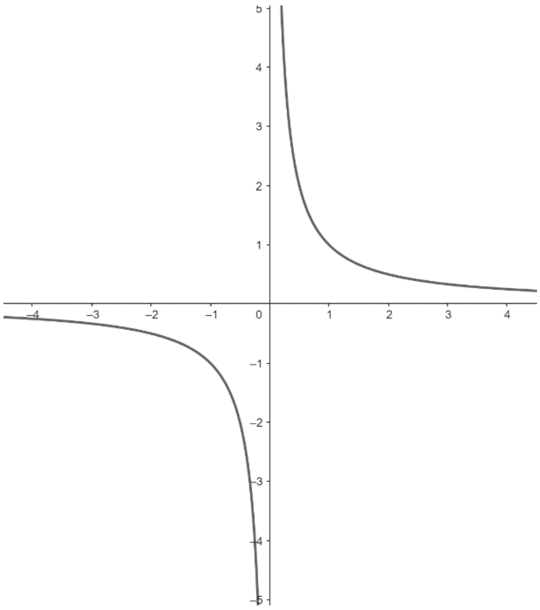
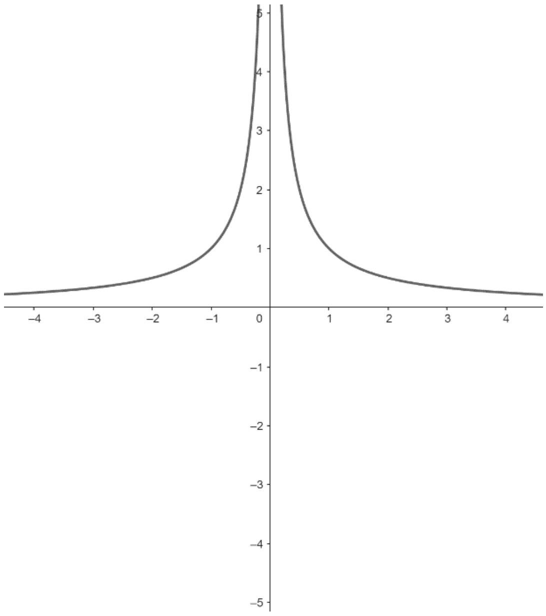
1) Which of the following graphs, correctly shows $y = \frac{|x|}{x^2}$?

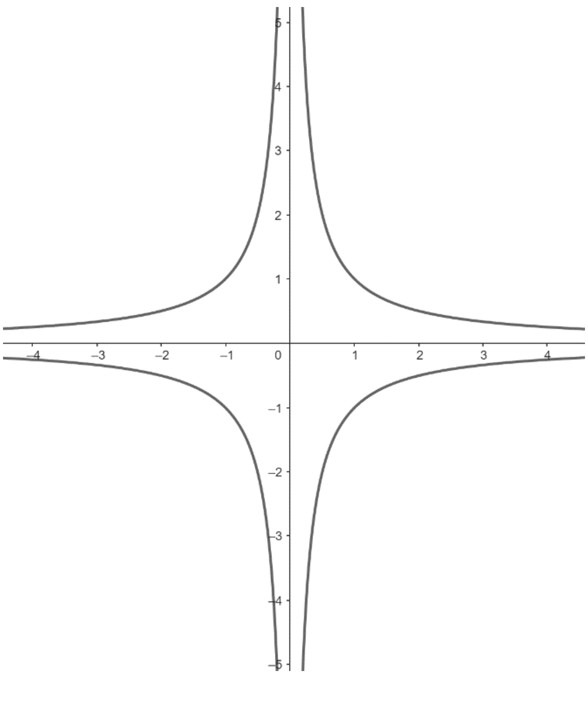
A.



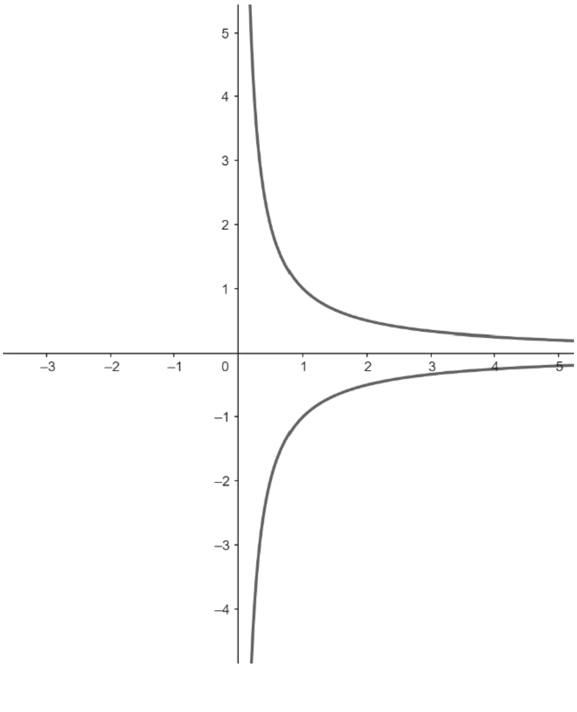
B.



C.



D.



2) A drawer contains 10 pairs of socks, each pair a different colour.

Michael decides to randomly select socks from the drawer without looking. What is the minimum number of socks he needs to pick to ensure he has at least one matching pair?

A. 2

B. 11

C. 12

D. 21

3) Which of the following is the constant term in the binomial expansion of $\left(2x - \frac{5}{x^3}\right)^{12}$?

A. $\binom{12}{3} 2^9 5^3$

B. $\binom{12}{9} 2^3 5^9$

C. $-\binom{12}{3} 2^9 5^3$

D. $-\binom{12}{9} 2^3 5^9$

4) What is the vector projection of $\underset{\sim}{a} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$ onto $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j}$?

A. $2\underset{\sim}{i} - 4\underset{\sim}{j}$

B. $4\underset{\sim}{i} - 2\underset{\sim}{j}$

C. $-2\underset{\sim}{i} + \underset{\sim}{j}$

D. $-4\underset{\sim}{i} + 2\underset{\sim}{j}$

5) What is the remainder when $2x^5 - 3x^3 + x^2 - 5$ is divided by $x + 2$?

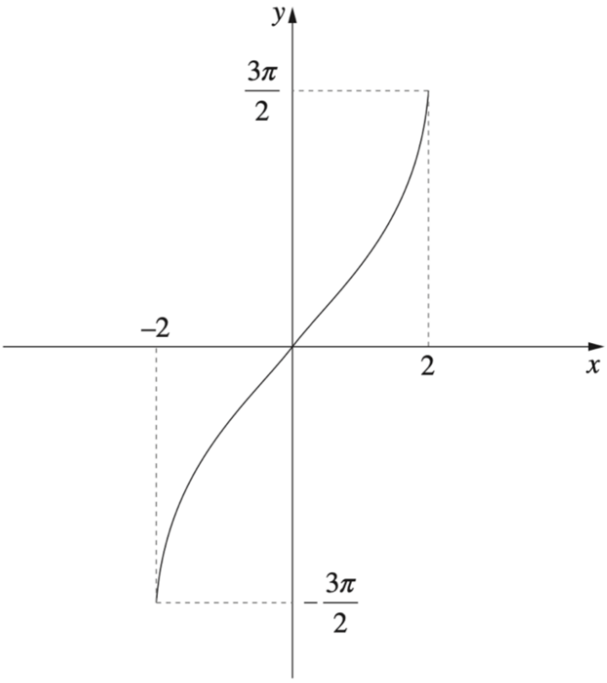
A. -41

B. -5

C. 39

D. 41

6) Which function best describes the following graph?



A. $y = 3\sin^{-1} 2x$

B. $y = \frac{3}{2}\sin^{-1} 2x$

C. $y = 3\sin^{-1}\frac{x}{2}$

D. $y = \frac{3}{2}\sin^{-1}\frac{x}{2}$

7) Which expression is equivalent to $\frac{\tan 2x - \tan x}{1 + \tan 2x \tan x}$?

A. $\tan x$

B. $\tan 3x$

C. $\frac{\tan 2x - 1}{1 + \tan 2x}$

D. $\frac{\tan x}{1 + \tan 2x \tan x}$

8) It is given that $\sin x = \frac{1}{4}$, where $\frac{\pi}{2} < x < \pi$.

What is the value of $\sin 2x$?

A. $-\frac{7}{8}$

B. $-\frac{\sqrt{15}}{8}$

C. $\frac{\sqrt{15}}{8}$

D. $\frac{7}{8}$

9) Sand pours onto the ground and forms a cone such that the height of the cone at time t seconds is h cm and the radius of the base is r cm.

It is given that $r = \sqrt{3}h$.

Sand is being poured onto the pile at a rate of 12 cm^3/s.

At which rate is the height increasing at the instant when the height is 12 cm?

A. $\frac{1}{24\pi}$ cm/s

B. $\frac{1}{36\pi}$ cm/s

C. $\frac{\pi}{24}$ cm/s

D. $\frac{\pi}{36}$ cm/s

10) At time t years the number, N of ants in an ant colony is given by $N = 500 - 400e^{-0.1t}$.

What is the rate of growth in ants/year when $t = 20$?

A. $40e^2$ ants/year

B. $-40e^2$ ants/year

C. $\frac{40}{e^2}$ ants/year

D. $-\frac{40}{e^2}$ ants/year

End of Section 1

SECTION II

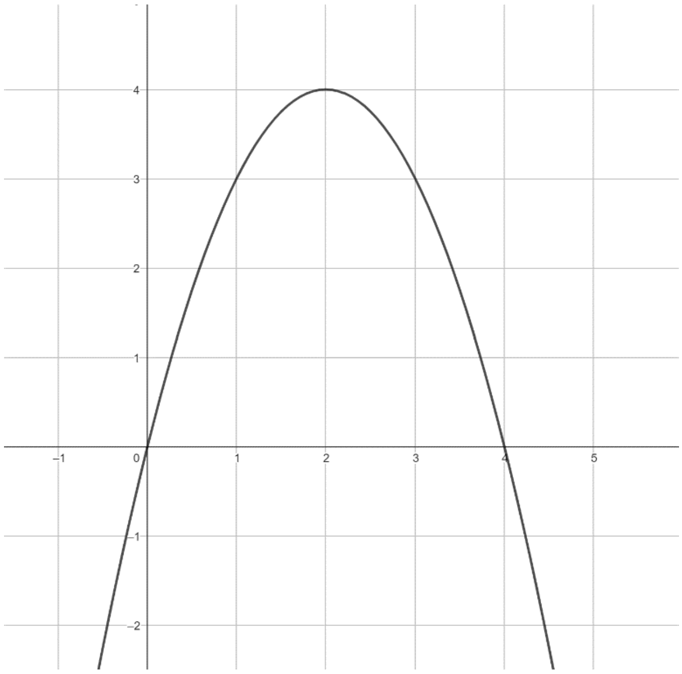
60 marks

Attempt Questions 11 - 15
Allow about 1 hour and 45 minutes on this section.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.
Answer each question in a separate answer booklet.

Question 11 (7 marks) Use the Question 11 BOOKLET **Marks**

(a) The graph shown is $f(x) = x(4 - x)$ **2**



In the Question 11 Answer Booklet, sketch the graph of $y = \sqrt{x(4 - x)}$

(b) Solve for x: $\frac{8-x}{x} \leq 1$ **3**

(c) (i) Eliminate t from the parametric equations: **1**

$$x = \cos t$$

$$y = \sin^2 t$$

(ii) Sketch the curve defined by the parametric equations in (i) **1**

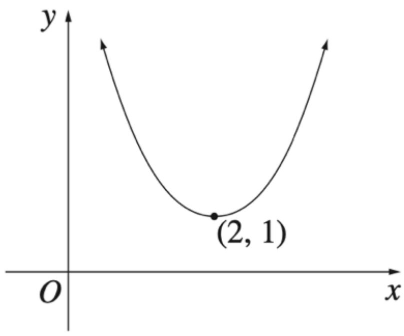
~End of Question 11~

Question 12 (10 marks) Use the Question 12 BOOKLET **Marks**

(a) (i) Write $\sqrt{3}\cos x - \sin x$ in the form $2\cos(x+\alpha)$, where $0 < \alpha < \frac{\pi}{2}$. **1**

(ii) Hence, or otherwise, solve $\sqrt{3}\cos x = 1 + \sin x$, where $0 < x < 2\pi$. **2**

(b) The graph of $f(x) = x^2 - 4x + 5$ is shown in the diagram.



(i) Explain why $f(x)$ does not have an inverse function. **1**

(ii) Sketch the graph of the inverse function $g^{-1}(x)$, of $g(x)$, where

$$g(x) = x^2 - 4x + 5, \ \ x \geq 2\,.$$ **1**

(iii) State the domain of $g^{-1}(x)$. **1**

(iv) Find an expression for $y = g^{-1}(x)$ in terms of x. **2**

(c) Prove that $\operatorname{cosec}\theta + \cot\theta = \cot\frac{\theta}{2}$ **2**

~End of Question 12~

Question 13 (15 marks) Use the Question 13 BOOKLET **Marks**

(a) Three boys and five girls are at a birthday party.

(i) The children are asked to sit in a circle for party games. In how many ways can the children be seated around the circle? **1**

(ii) For the *'Pass the Parcel'* game, the children remain in the circle, but two of the boys are asked not to sit together. In how many ways may this occur? **2**

(b) (i) Show that $\frac{d}{dx}[\sin^{-1}(2x-1)] = \frac{1}{\sqrt{x-x^2}}$ **2**

(ii) Hence, or otherwise show that

$$\int_{\frac{3}{4}}^{1} \frac{dx}{\sqrt{x-x^2}} = \frac{\pi}{3}$$ **2**

(c) Using the substitution $x = 2\cos\theta$, evaluate:

$$\int_{1}^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$$ **4**

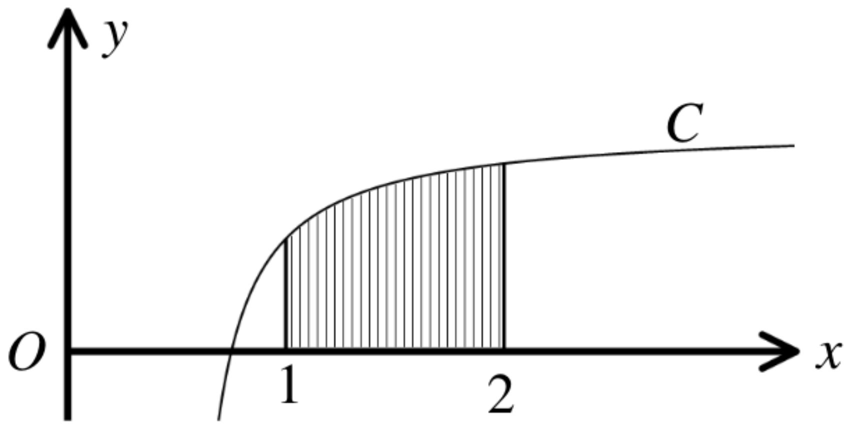
Leaving your answer in simplest exact form.

Question 13 is continued on next page...

Question 13 continued... **Marks**

(d) The figure below shows part of a graph of the curve C with the equation

$$y = 2 - \frac{1}{2x - 1} \qquad \text{, where } x \neq \frac{1}{2}$$



The shaded region is bounded by C and the lines $x = 1$, $x = 2$ and the x-axis is rotated about the x-axis, forming a solid of revolution.

Show that the volume of the solid is

$$\pi\left(\frac{13}{3} - 2\ln 3\right) \text{units}^3$$ **4**

~End of Question 13~

Question 14 (13 marks) Use the Question 14 BOOKLET **Marks**

(a) Prove by mathematical induction that for integers $n \geq 1$,

$$\frac{3}{1 \times 2 \times 2} + \frac{4}{2 \times 3 \times 2^2} + \frac{5}{3 \times 4 \times 2^3} + \cdots + \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(n+1)2^n}$$ **3**

(b) A spherical weather balloon is being inflated from empty using a container of helium, so that its volume is increasing at a constant rate of $0.5 \text{ m}^3/\text{s}$.
(Assume the balloon maintains a spherical shape throughout inflation.)

(i) By considering the volume after t seconds, show that the radius at time t is given by $r = \sqrt[3]{\frac{3t}{8\pi}}$ m **2**

(ii) Show that the rate of increase of its surface area after 8 seconds is $\sqrt[3]{\frac{\pi}{3}} \text{ m}^2/\text{s}$. **3**

Question 14 is continued on next page...

Question 14 continued... **Marks**

(c) Water is being heated in a kettle.

At time t seconds, the temperature of the water is T^oC.

The rate of increase of the temperature of the water at any time after the kettle is switched on, is modelled by the equation $\frac{dT}{dt} = k(120 - T)$, where k is a positive constant.

The temperature of the water is at 20^o C when the kettle is switched on.

(i) Show that $T = 120 - 100e^{-kt}$ is both a solution to the differential equation and satisfies the initial conditions. **2**

(ii) When the temperature of the water reaches 100^o C, the kettle switches off. If it takes 10 seconds for the temperature to reach 30^o C, once the kettle is switched on, find how long it takes for the kettle to switch off, to the nearest second. **3**

~End of Question 14~

Question 15 (15 marks) Use the Question 15 BOOKLET **Marks**

(a) The roots of $2x^3 - 6x^2 - 8x + 12 = 0$ are α, β and γ.

Find the value of $(\alpha + 2)(\beta + 2)(\gamma + 2)$. **3**

(b) When $\left(1 - \frac{3}{2}x\right)^p$ is expanded in ascending powers of x, the coefficient of x is -24.

By considering the expansion of $(1 + x)^P$, find the value of p. **2**

(c) Show that ${}^{n-1}C_{k-1} + {}^{n-1}C_k = \frac{n!}{(n-k)!\ k!}$ **2**

Question 15 is continued on next page...

(d) Consider the points $P(a, 2a)$, $Q(-a, 5a)$, $R(3a, 4a)$ and $S(9a, 12a)$, where a is a positive real number.

(i) Express vector $\overrightarrow{PQ}$ in the form $p\underset{\sim}{i} + q\underset{\sim}{j}$ where p and q are real numbers. **1**

(ii) Given the length of the projection of vector $\overrightarrow{PQ}$ onto vector $\overrightarrow{RS}$ is 12, find the value of a. **3**

(e) A quadrilateral $ABCD$ has $|AB| = |BC|$ and $|CD| = |DA|$.

Let M be the point of intersection of AC and BD. M bisects AC.

(i) Show that $\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BC} - \frac{1}{2}\overrightarrow{AB}$ **2**

(ii) Using (i), prove that the diagonals of a kite are perpendicular. **2**

~End of Question 15~

End of Section II

Multiple Choice

Question 1

If $y = \frac{|x|}{x^2}$, then $y = \begin{cases} \frac{-x}{x^2}, & x < 0 \\ \frac{x}{x^2}, & x \geq 0 \end{cases}$

$\therefore y = \begin{cases} \frac{-1}{x}, & x < 0 \\ \frac{1}{x}, & x \geq 0 \end{cases}$

For $y = \frac{-1}{x}$ where $x < 0$, the required branch of the hyperbola is in quadrant 2.
For $y = \frac{1}{x}$ where $x \geq 0$, the branch of the required hyperbola is in quadrant 1.
$\therefore$ ***B***

Question 2

Pigeonholes **:** The 10 different colours of socks
Pigeons**:** The individual socks Michael picks.

If Michael picks 10 socks, it's possible that he might pick one sock of each colour (i.e., 1 sock from each of the 10 different colours). In this case, he still wouldn't have a matching pair.

However, if he picks one more sock (the 11th sock), he will definitely have to pick a sock that matches one of the socks he has already picked, because there are only 10 colours available.

Thus, by the pigeonhole principle, **after picking 11 socks**, he is guaranteed to have at least one matching pair.

$\therefore$ ***B***

Question 3

It will be the 4th term of the expansion that will be a constant.
The 4th term will be $\binom{12}{3}(2x)^9\left(-\frac{5}{x^3}\right)^3$
Which simplifies to $-\binom{12}{3}2^9 5^3$
Therefore answer is C.

Question 4

$$a \cdot b = (-3)(2) + (4)(-1) = -6 - 4 = -10$$

$$b \cdot b = (2)(2) + (-1)(-1) = 4 + 1 = 5$$

$$\text{proj}_b a = 5 - 10b = -2b$$

$$\text{proj}_b a = -2(2i - j) = -4i + 2j$$

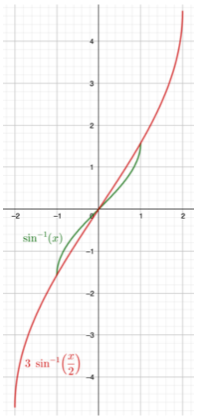
Answer **D**

Question 5

$$P(x) = 2x^5 - 3x^3 + x^2 - 5$$
$$P(-2) = 2(-2)^5 - 3(-2)^3 + (-2)^2 - 5 = -41$$

Answer : A

6 The given graph is is $y = \sin^{-1} x$ stretched vertically by a factor of 3, and horizontally by a factor of 2.



Ans: C

7 $\frac{\tan A - \tan B}{1 + \tan A \tan B} = \tan(A - B)$

$\frac{\tan 2x - \tan x}{1 + \tan 2x \tan x} = \tan(2x - x) = \tan x$ **Ans: A**

8 $\sin x = \frac{1}{4}$, where $\frac{\pi}{2} < x < \pi \quad \Rightarrow \quad \cos x = -\frac{\sqrt{15}}{4}$

$\sin 2x = 2 \sin x \cos x$

$= 2 \times \frac{1}{4} \times \frac{-\sqrt{15}}{4} = -\frac{\sqrt{15}}{8}$ **Ans: B**

9

Given $r = \sqrt{3}h$

$V = \frac{1}{3}\pi(\sqrt{3}h)^2 h = \frac{3\pi h^3}{3} = \pi h^3$

$\frac{dV}{dh} = 3\pi h^2$

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} = 12 \times \frac{1}{3\pi h^2}$

When $h = 12cm$

$\frac{dh}{dt} = 12 \times \frac{1}{3\pi 12^2} = \frac{1}{36\pi} cm/s$

∴***Answer:***

B

MC Q10

$N = 500 - 400e^{-0.1t}$

$\frac{dN}{dt} = -400(-0.1)e^{-0.1t} = 40e^{-0.1t}$

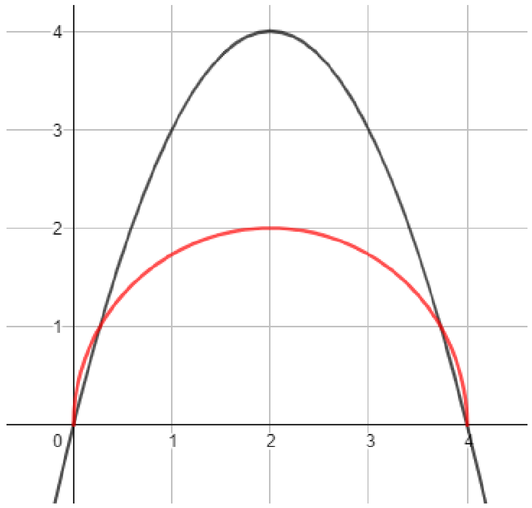
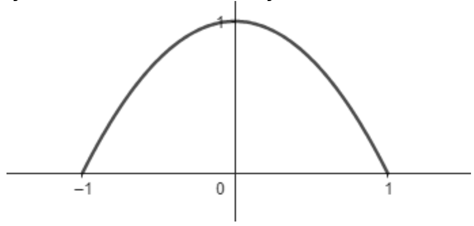
$at\ t = 20, \frac{dN}{dt} = 40e^{-0.1(20)}$

$\frac{dN}{dt} = 40e^{-2}$

∴***Answer: C***

Outcomes Addressed in this Question

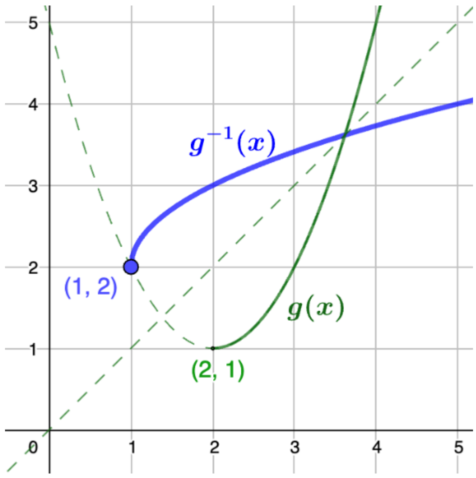
ME11-1 uses algebraic and graphical concepts in the modelling and solving of problems involving functions and their inverses
ME11-2 manipulates algebraic expressions and graphical functions to solve problems

Outcome	Solutions	Marking Guidelines
ME11-2, ME11-1	(a) 	2 marks : correct solution 1 mark: correct shape, maximum & domain or identifies required detail around y = 1
ME11-2	(b) $\frac{8-x}{x} \leq 1\,,\quad x \neq 0$ $x^2 \times \frac{8-x}{x} \leq x^2 \times 1$ $x(8-x) - x^2 \leq 0$ $x(8-x-x) \leq 0$ $x(8-2x) \leq 0$ $x(4-x) \leq 0$ From the diagram in (a), $x \leq 0$ and $x \geq 4$ But, $x \neq 0$ $\therefore$ (-∞, 0) ∪ [4, ∞)	3 marks : correct solution 2 marks : two of demonstrates an appropriate method of solution, demonstrates correct method of solving quadratic inequality or identifies $x \neq 0$ 1 mark: one of above
ME11-2	(c) (i) Given $x = \cos t$ and $y = \sin^2 t$, as $\sin^2 t + \cos^2 t = 1,$ $y + x^2 = 1$ $\therefore y = 1 - x^2$	1 mark : correct answer
ME11-2, ME11-1	(ii) Curve is the parabola $y = 1 - x^2$, but since $x = \cos t,$ then $-1 \leq x \leq 1$ and since $y = \sin^2 t$ then $0 \leq y \leq 1.$ 	1 mark : correct solution

Outcomes Addressed in this Question

ME11-1 uses algebraic and graphical concepts in the modelling and solving of problems involving functions and their inverses
ME12-3 applies advanced concepts and techniques in simplifying expressions involving compound angles and solving trigonometric equations

Part / Outcome	Solutions	Marking Guidelines
(a)(i) **12-3**	$2\cos(x+\alpha) = \sqrt{3}\cos x - \sin x$ $\cos(x+\alpha) = \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$ $\cos x\cos\alpha - \sin x \sin\alpha = \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$ $\cos\alpha = \frac{\sqrt{3}}{2}, \quad \sin\alpha = \frac{1}{2} \quad \Rightarrow \quad \tan\alpha = \frac{1}{\sqrt{3}}$ $\alpha = \frac{\pi}{6}$ $\therefore\ \sqrt{3}\cos x - \sin x = 2\cos\left(x+\frac{\pi}{6}\right)$	**1 mark** – Correct solution **It's all about α, really**
(a)(ii) **12-3**	$\sqrt{3}\cos x = 1 + \sin x$ $\sqrt{3}\cos x - \sin x = 1$ $2\cos\left(x+\frac{\pi}{6}\right) = 1$ $\cos\left(x+\frac{\pi}{6}\right) = \frac{1}{2}$ $x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$ $x = \frac{\pi}{6}, \frac{3\pi}{2} \qquad (0 < x < 2\pi)$ *question 12 continued…*	**2 marks** – Correct solution **1 mark** – Substantially correct *Note: many students attempted to the general solutions formula. Most of those did not find two correct answers. There is a reason it was taken out of the syllabus*

<table>
<tr><td>(b)(i)
11-1</td><td>$f(x)$ does not have an inverse function as it is not 1 – 1.
… or… it fails the horizontal line test.</td><td>1 mark – Correct solution</td></tr>
<tr><td>(b)(ii)
11-1</td><td></td><td>1 mark – Correct shape and position of graph
Note: a LOT of answers had carelessly drawn graphs. It is a half parabola, and should be vertical at the point (1, 2).
Some graphs were drawn in such a way that it was <u>clearly not</u> parabolic. This cost marks</td></tr>
<tr><td>(b)(iii)
11-1</td><td>$g^{-1}(x)$ has domain $x \geq 1$ or $[1, \infty)$</td><td>1 mark – Correct solution</td></tr>
<tr><td>(b)(iv)
11-1</td><td>$$\begin{aligned} x &= y^2 - 4y + 5 \\ y^2 - 4y &= x - 5 \\ y^2 - 4y + 4 &= x - 5 + 4 \\ (y-2)^2 &= x - 1 \\ y - 2 &= \pm\sqrt{x-1} \quad \text{(take positive square root)} \\ \therefore \; g^{-1}(x) &= 2 + \sqrt{x-1} \end{aligned}$$</td><td>2 marks – Correct solution

1 mark – partial progress towards correct solution</td></tr>
<tr><td>(c)
12-3</td><td>$using \;\; t = \tan\frac{\theta}{2}$,
$$\begin{aligned} \operatorname{cosec}\theta + \cot\theta &= \frac{1}{\sin\theta} + \frac{1}{\cot\theta} \\ &= \frac{1+t^2}{2t} + \frac{1-t^2}{2t} \\ &= \frac{2}{2t} = \frac{1}{t} \\ &= \frac{1}{\tan\frac{\theta}{2}} \\ &= \cot\frac{\theta}{2} \end{aligned}$$
Or…
$$\begin{aligned} &\frac{1}{\sin\theta} + \frac{1}{\cot\theta} \\ &= \frac{1+\cos\theta}{\sin\theta} \\ &= \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \\ &= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \\ &= \cot\frac{\theta}{2} \end{aligned}$$</td><td>2 marks – Correct solution

1 mark – partial progress towards correct solution
Correctly expresses cosec θ or cot θ in terms of t
Note: getting to $\frac{1+\cos\theta}{\sin\theta}$ was not enough for first mark</td></tr>
</table>

Outcomes Addressed in this Question
ME11-5 uses concepts of permutations and combinations to solve problems involving counting or ordering **ME12-4** uses calculus in the solution of applied problems, including differential equations and volumes of solids of revolution

Question 13	Solutions	Marking Guidelines
(a) ME11-5 **(i)**	$7! = 5040$	**1 mark** Correct solution
(ii)	Total ways two boys not seated together $= 7! - (6! \times 2)$ $= 3600$	**2 marks** Correct solution **1 mark** Makes some progress towards the correct solution.
(b) ME12-4 **(i)**	$\frac{d}{dx}[\sin^{-1}(2x-1)]$ $let\ u = 2x - 1$ $= \frac{1}{\sqrt{1-u^2}} \times \frac{d}{dx}(2x-1)$ $= \frac{2}{\sqrt{1-(2x-1)^2}}$ $= \frac{2}{1-(4x^2-4x+1)}$ $= \frac{2}{\sqrt{4x-4x^2}}$ $= \frac{1}{\sqrt{x-x^2}}$	**2 marks** Correct solution with complete setting out. **1 mark** ONE error OR some progress towards the correct solution.

(ii)	$$\int_{\frac{3}{4}}^{1}[\sin^{-1}(2\mathrm{x}-1)]dx$$ $$\left[\sin^{-1}(2\mathrm{x}-1)\right]_{\frac{3}{4}}^{1}$$ $$= sin^{-1}(1) - sin^{-1}\left(\frac{1}{2}\right)$$ $$= \frac{\pi}{2} - \frac{\pi}{6}$$ $$= \frac{\pi}{3}$$	**2 marks** Correct solution AND shows substitution process. **1 mark** Some progress towards the correct solution OR single error.
(c) ME12-4	$$x = 2\cos\theta$$ $$\frac{dx}{d\theta} = -2\sin\theta$$ $$dx = -2\sin\theta\, d\theta$$ When $x = 1, \theta = \frac{\pi}{3}$ When $x = \sqrt{2}, \theta = \frac{\pi}{4}$ $$\int_{1}^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}dx$$ $$= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{(2\cos\theta)^2\sqrt{4-(2\cos\theta)^2}} \times -2\sin\theta\, d\theta$$ $$= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{-2\sin\theta}{4\cos^2\theta\sqrt{4-4\cos^2\theta}}\, d\theta$$ $$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sin\theta}{4\cos^2\theta\sqrt{4(1-\cos^2\theta)}}\, d\theta$$ $$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sin\theta}{4\cos^2\theta\sqrt{4\sin^2\theta}}\, d\theta$$ $$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sin\theta}{8\cos^2\theta\sin\theta}\, d\theta$$	**4 marks** Complete solution shown. **3 marks** Makes significant progress towards the correct solution with ONE error **2 marks** Makes some progress towards the correct solution with TWO errors. **1 mark** Makes limited progress towards the correct solution with THREE errors.

	$= \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2\theta \, d\theta$ $= \frac{1}{4}\left[\tan\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $= \frac{1}{4}[\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)]$ $= \frac{1}{4}\left(\sqrt{3} - 1\right)$	
(d) **ME12-4**	$y^2 = \left(2 - \frac{1}{2x-1}\right)^2$ $y^2 = 4 - \frac{4}{2x-1} + \frac{1}{(2x-1)^2}$ $V = \pi\int_1^2 4 - \frac{4}{2x-1} + \frac{1}{(2x-1)^2} \, dx$ $= \pi\left[4x - 2\ln\lvert 2x-1\rvert - \frac{1}{2}(2x-1)^{-1}\right]_1^2$ $= \pi[(8 - 2\ln 3 - \frac{1}{6}) - (4 - 2\ln 1 - \frac{1}{2})]$ $= \pi(8 - 2\ln 3 - \frac{1}{6} - 4 + \frac{1}{2})$ $= \pi(\frac{13}{3} - 2\ln 3)$	**4 marks** Writes an expression for volume of the solid and complete solution shown. **3 marks** Makes significant progress towards the correct solution with ONE error **2 marks** Makes some progress towards the correct solution with TWO errors. **1 mark** Makes limited progress towards the correct solution with THREE errors.

Solutions and Marking Guidelines

Outcomes Addressed in this Question

ME12-1 applies techniques involving proof or calculus to model and solve problems

Solutions	Marking Guidelines
Question 14 a	
Prove $n = 1$ is true $LHS = \frac{3}{1\times2\times2} = \frac{3}{4}$ $RHS = 1 - \frac{1}{2\times2} = \frac{3}{4}$ Therefore n = 1 is true. Assume $n = k$ is true $\frac{3}{1\times2\times2} + \frac{4}{2\times3\times2^2} + \cdots + \frac{k+2}{k(k+1)2^k} = 1 - \frac{1}{(k+1)2^k}$ Prove $n = k + 1$ is true, i.e. $\frac{3}{1\times2\times2} + \cdots + \frac{k+2}{k(k+1)2^k} + \frac{k+3}{(k+1)(k+2)2^{k+1}} = 1 - \frac{1}{(k+2)2^{k+1}}$ $LHS = \frac{3}{1\times2\times2} + \cdots + \frac{k+2}{k(k+1)2^k} + \frac{k+3}{(k+1)(k+2)2^{k+1}}$ $= 1 - \frac{1}{(k+1)2^k} + \frac{k+3}{(k+1)(k+2)2^{k+1}}$ $= 1 - \left[\frac{2(k+2)}{(k+1)(k+2)2^{k+1}} - \frac{k+3}{(k+1)(k+2)2^{k+1}}\right]$ $= 1 - \frac{2(k+2) - k - 3}{(k+1)(k+2)2^{k+1}}$ $= 1 - \frac{\cancel{k+1}}{\cancel{(k+1)}(k+2)2^{k+1}}$ $= 1 - \frac{1}{(k+2)2^{k+1}}$ $= RHS$ $\therefore n = k + 1$ is true whenever $n = k$ is true. $\therefore$ As it is true for $n = 1$, by induction, the result is true for $n \geq 1$.	**3 marks** – Correct solution **2 marks** – Substantially correct solution. **1 mark** – some correct working towards correct solution

Question 14 b **(i)** $\frac{dV}{dt} = 0.5$ $V = 0.5t + C$ When $t = 0, V = 0$ $V = \frac{t}{2}$ $V = \frac{4}{3}\pi r^3$ $\frac{t}{2} = \frac{4}{3}\pi r^3$ $3t = 8\pi r^3$ $r^3 = \frac{3t}{8\pi}$ $r = \sqrt[3]{\frac{3t}{8\pi}}$ $\therefore r = \sqrt[3]{\frac{3t}{8\pi}}$ m as required.	**2 marks** – Correct solution **1 mark** – Substantially correct solution.
(ii) $A = 4\pi r^2$ $\frac{dA}{dr} = 8\pi r$ $V = \frac{4}{3}\pi r^3$ $\frac{dV}{dr} = 4\pi r^2$ $\frac{dV}{dt} = 0.5$ $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$ $\frac{dA}{dt} = 8\pi r \times \frac{1}{4\pi r^2} \times 0.5$ $\frac{dA}{dt} = 2 \times \frac{1}{r} \times 0.5 = \frac{1}{r} = \frac{1}{\sqrt[3]{\frac{3t}{8\pi}}}$ When $t = 8\ seconds$	**3 marks** – Correct solution **2 marks** – Substantially correct solution. **1 mark** – some correct working towards correct solution

$$\frac{dA}{dt} = \frac{1}{\sqrt[3]{\frac{3\times 8}{8\pi}}} = \frac{1}{\sqrt[3]{\frac{3}{\pi}}} = \sqrt[3]{\frac{\pi}{3}}\ \mathrm{m^2/s}$$ **Alternate method:** $r = \sqrt[3]{\frac{3t}{8\pi}}$ $r = \left(\frac{3t}{8\pi}\right)^{\frac{1}{3}}$ $r = \left(\frac{3}{8\pi}\right)^{\frac{1}{3}} \times t^{\frac{1}{3}}$ $\frac{dr}{dt} = \left(\frac{3}{8\pi}\right)^{\frac{1}{3}} \times \frac{t^{-\frac{2}{3}}}{3}$ $A = 4\pi r^2$ $\frac{dA}{dr} = 8\pi r$ $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ $= 8\pi r \times \left(\frac{3}{8\pi}\right)^{\frac{1}{3}} \times \frac{t^{-\frac{2}{3}}}{3}$ $= 8\pi\left(\left(\frac{3t}{8\pi}\right)^{\frac{1}{3}}\right) \times \left(\left(\frac{3}{8\pi}\right)^{\frac{1}{3}} \times \frac{t^{-\frac{2}{3}}}{3}\right)$ When $t = 8$ $\frac{dA}{dt} = 8\pi\left(\left(3\frac{8}{8\pi}\right)^{\frac{1}{3}}\right) \times \left(\left(\frac{3}{8\pi}\right)^{\frac{1}{3}} \times \frac{(8)^{-\frac{2}{3}}}{3}\right)$ $= 8\pi \times \left(\frac{3}{\pi}\right)^{\frac{1}{3}} \times \frac{1}{2}\left(\frac{3}{\pi}\right)^{\frac{1}{3}} \times \frac{1}{12}$ $= \frac{\pi}{3}\left(\frac{3}{\pi}\right)^{\frac{2}{3}}$ $= \frac{\pi}{3}\left(\frac{\pi}{3}\right)^{-\frac{2}{3}}$ $= \left(\frac{\pi}{3}\right)^{\frac{1}{3}}$ $= \sqrt[3]{\frac{\pi}{3}}$	**2 marks** – Correct solution **1 mark** – Substantially correct solution.
Question 14 c **(i)** $T = 120 - 100e^{-kt}$ $\frac{dT}{dt} = 100ke^{-kt}$ $= k(100e^{-kt} - 120 + 120)$ $= k(120 - T)$ when $t = 0$, $T = 120 - 100e^{0}$ $= 120 - 100$ $= 20$ **(ii)**	**3 marks** – Correct solution **2 marks** – Substantially correct solution. **1 mark** – some correct working towards correct solution

when $t = 10$ $T = 30$

$$30 = 120 - 100e^{-10k}$$
$$e^{-10k} = \frac{9}{10}$$
$$-10k = \ln\frac{9}{10}$$
$$k = -\frac{1}{10}\ln\frac{9}{10}$$
$$= 0.01053605157...$$

when $T = 100$, $100 = 120 - 100e^{-kt}$

$$100e^{-kt} = 20$$
$$e^{-kt} = \frac{1}{5}$$
$$-kt = \ln\frac{1}{5}$$
$$t = -\frac{1}{k}\ln\frac{1}{5}$$
$$t = 152.7553185...$$

The kettle switches off after 153 seconds

Question 15

a)

$$\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{-6}{2} = 3$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{-8}{2} = -4$$
$$\alpha\beta\gamma = -\frac{d}{a} = -\frac{12}{2} = -6$$

$(\alpha + 2)(\beta + 2)(\gamma + 2) = (\alpha\beta + 2\alpha + 2\beta + 4)(\gamma + 2)$
$= (\alpha\beta\gamma + 2\alpha\gamma + 2\beta\gamma + 4\gamma +$
$2\alpha\beta + 4\alpha + 4\beta + 8)$
$= \alpha\beta\gamma + 2(\alpha\gamma + \beta\gamma + \alpha\beta) +$
$4(\gamma + \alpha + \beta) + 8$
$= -6 + 2(-4) + 4(3) + 8$
$= 6$

3 marks – Correct solution
2 marks – Substantially correct solution.
1 mark – some correct working towards correct solution

b)

$$\left(1 - \frac{3}{2}x\right)^p = 1^p + {}^pC_1(1)^{p-1}\left(-\frac{3x}{2}\right)$$
$$= 1 + p\left(-\frac{3x}{2}\right)$$
$$= 1 + \left(-\frac{3p}{2}\right)x$$

Coefficient of x is $-\frac{3p}{2}$

$$-\frac{3p}{2} = -24$$
$$p = 16$$

2 marks – Correct solution
1 mark – Substantially correct solution.

c) $$\text{LHS} = {}^{n-1}C_{k-1} + {}^{n-1}C_k = \frac{(n-1)!}{(n-1-(k-1))!\ (k-1)!} + \frac{(n-1)!}{(n-1-k)!\ k!}$$ $$= \frac{(n-1)!}{(n-k)!\ (k-1)!} + \frac{(n-1)!}{(n-1-k)!\ k!}$$ $$= \frac{(n-1)!}{(n-k)(n-k-1)!\ (k-1)!} + \frac{(n-1)!}{(n-1-k)!\ k(k-1)!}$$ $$= \frac{(n-1)!}{(n-k)(n-k-1)!\ (k-1)!} + \frac{(n-1)!}{(n-1-k)!\ k\,(k-1)!}$$ $$= \frac{k\,(n-1)!}{k\,(n-k)\,(n-k-1)!\ (k-1)!} + \frac{(n-k)\,(n-1)!}{(n-k)\ (n-1-k)!\ k\,(k-1)!}$$ $$= \frac{k\,(n-1)! + (n-k)\,(n-1)!}{k\,(n-k)\,(n-k-1)!\ (k-1)!}$$ $$= \frac{k\,(n-1)! + (n-k)\,(n-1)!}{k!\ (n-k)!}$$ $$= \frac{k\,(n-1)! + n\,(n-1)! - k(n-1)!}{k!\ (n-k)!}$$ $$= \frac{n\,(n-1)!}{k!\ (n-k)!}$$ $$= \frac{n\,!}{k!\ (n-k)!} = \text{RHS}$$	**2 marks** – Correct solution **1 mark** – Substantially correct solution.
d i) $$\overrightarrow{PQ} = (-a-a)\boldsymbol{i} + (5a-2a)\boldsymbol{j} = -2a\boldsymbol{i} + 3a\boldsymbol{j}$$	**1 mark** – Correct solution
d ii) $\overrightarrow{RS} = (9a-3a)\boldsymbol{i} + (12a-4a)\boldsymbol{j} = 6a\boldsymbol{i} + 8a\boldsymbol{j}$	**3 marks** – Correct

$$\overrightarrow{PQ} \cdot \overrightarrow{RS} = \begin{pmatrix} -2a \\ 3a \end{pmatrix} \cdot \begin{pmatrix} 6a \\ 8a \end{pmatrix}$$ $$= -12a^2 + 24a^2 = 12a^2$$ $$\left|\overrightarrow{RS}\right| = \sqrt{(6a)^2 + (8a)^2}$$ $$= \sqrt{100\,a^2} = 10a$$ The length of Proj of $\overrightarrow{PQ}$ onto $\overrightarrow{RS}$: $$\frac{\overrightarrow{PQ} \cdot \overrightarrow{RS}}{\left|\overrightarrow{RS}\right|} = \frac{12a^2}{10a}$$ $$\frac{12a^2}{10a} = 12$$ $$6a = 60$$ $$\therefore \quad a = 10$$	solution **2 marks** – Substantially correct solution. **1 mark** – some correct working towards correct solution
e) i) $$\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM}$$ $$= \overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}$$ $$= -\overrightarrow{AB} + \tfrac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right)$$ $$= -\tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC}$$	**2 marks** – Correct solution **1 mark** – Substantially correct solution.
e) ii) For $\overrightarrow{AC} \perp \overrightarrow{BM}$ ----> $\overrightarrow{AC} \cdot \overrightarrow{BM} = 0$ $$\overrightarrow{AC} \cdot \overrightarrow{BM} = \left(\overrightarrow{AB} + \overrightarrow{BC}\right) \cdot \left(-\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}\right)$$ $$= \left(-\frac{1}{2}\overrightarrow{AB} \cdot \overrightarrow{AB}\right) + \left(\overrightarrow{AB} \cdot \frac{1}{2}\overrightarrow{BC}\right) + \left(-\frac{1}{2}\overrightarrow{AB} \cdot \overrightarrow{BC}\right) + \left(\overrightarrow{BC} \cdot \frac{1}{2}\overrightarrow{BC}\right)$$ $$= -\frac{1}{2}\left|\overrightarrow{AB}\right|^2 + \frac{1}{2}\left|\overrightarrow{BC}\right|^2$$ $$= -\tfrac{1}{2}\left|\overrightarrow{AB}\right|^2 + \tfrac{1}{2}\left|\overrightarrow{AB}\right|^2 \qquad (\text{as } \|AB\| = \|BC\|)$$ $$= 0$$ $$\therefore \quad \overrightarrow{AC} \perp \overrightarrow{BM}$$ $\therefore$ diagonals of a kite are perpendicular	**2 marks** – Correct solution **1 mark** – Substantially correct solution.